Greedy Approximations & LP duality

Big Picture

many discrete problems are hard

what next?

- Approximations
- continuous relaxations

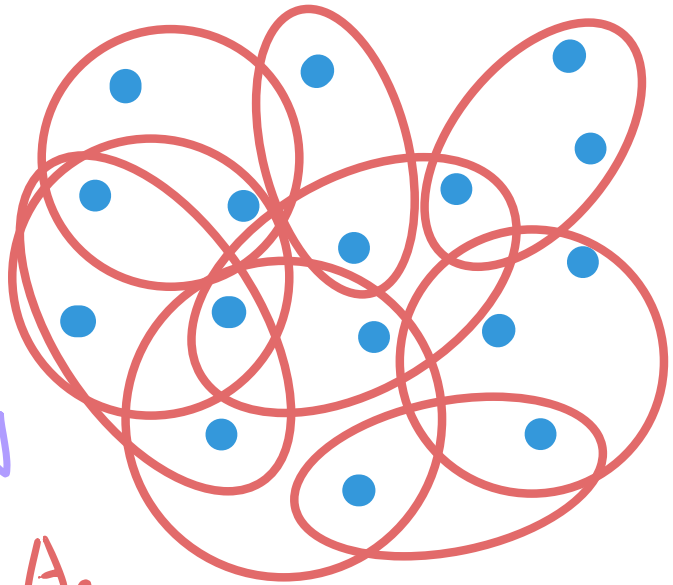
Part I: Maximum Coverage

Maximum Coverage

m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, A_2, \sim, A_n \subseteq [m]$

cardinality constraint $k \in \mathbb{N}$



Goal: select k sets $A_{e_1}, \sim, A_{e_k}$

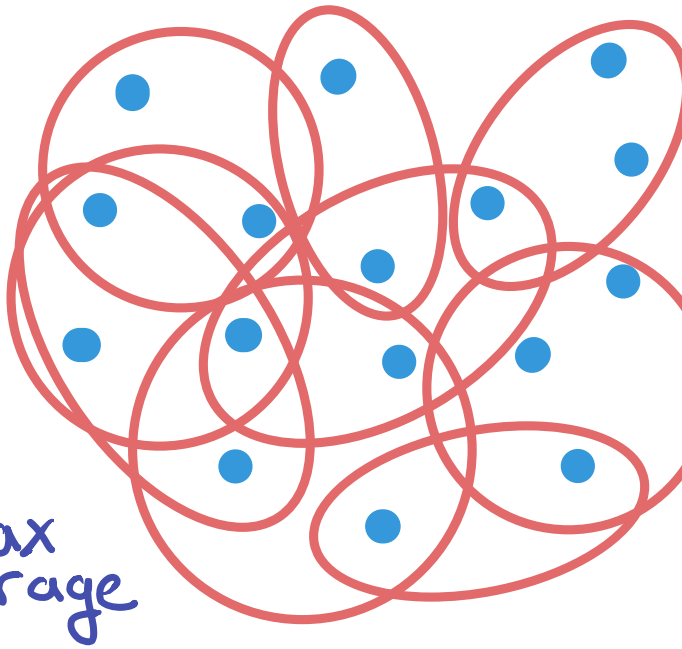
maximizing the $|A_{e_1} \cup A_{e_2} \cup \sim, A_{e_k}|$

NP-Hard via set cover

next time

m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, \ldots, A_n \subseteq [m]$



NP-Hard

Set Cover (next time)

minimize # sets to cover all points

NP-Hard (via SAT)

LP + randomized rounding

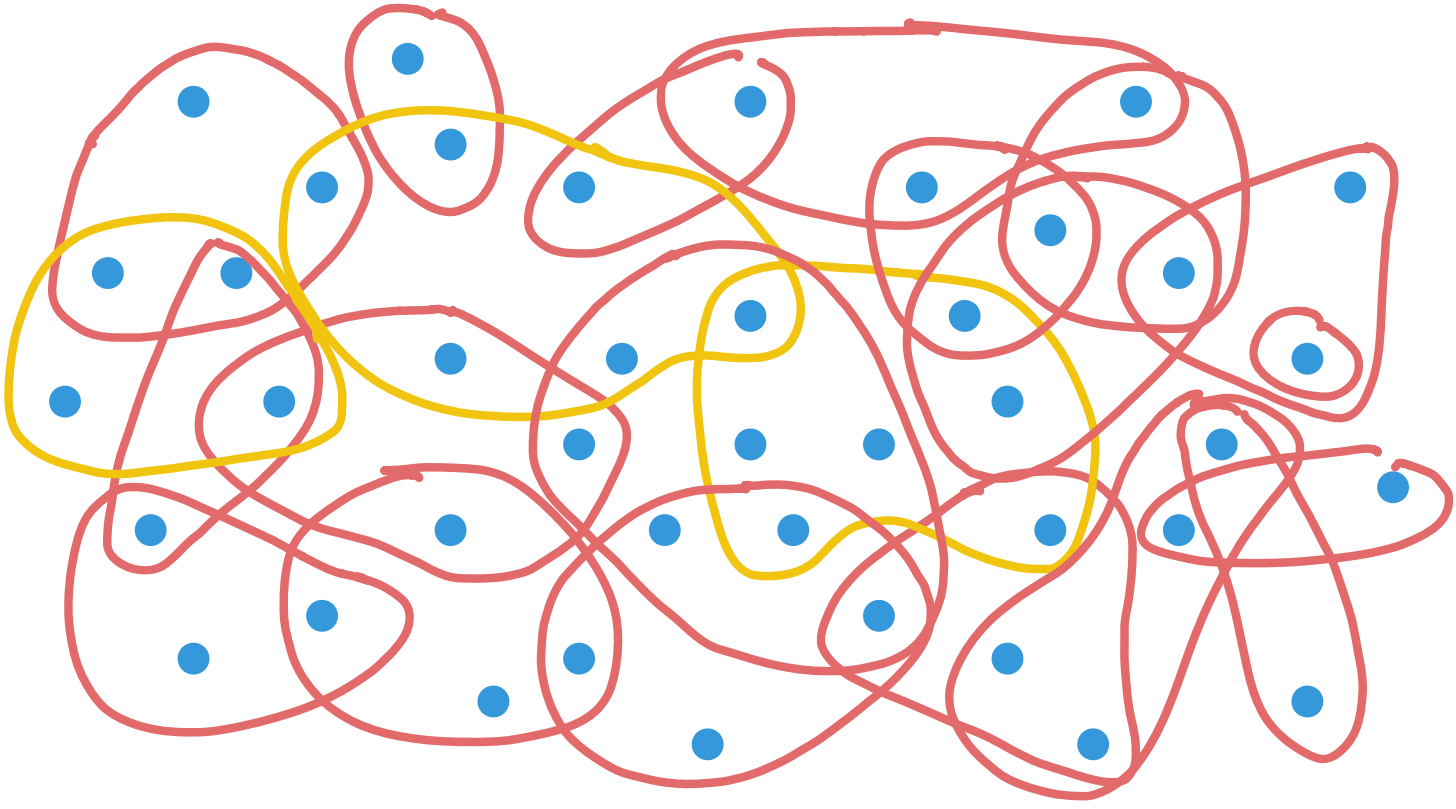
Max Coverage

maximize # points covered w/ $\leq k$ sets (packing constraint)

(why?) $\Rightarrow$ NP-Hard

greedy algorithm

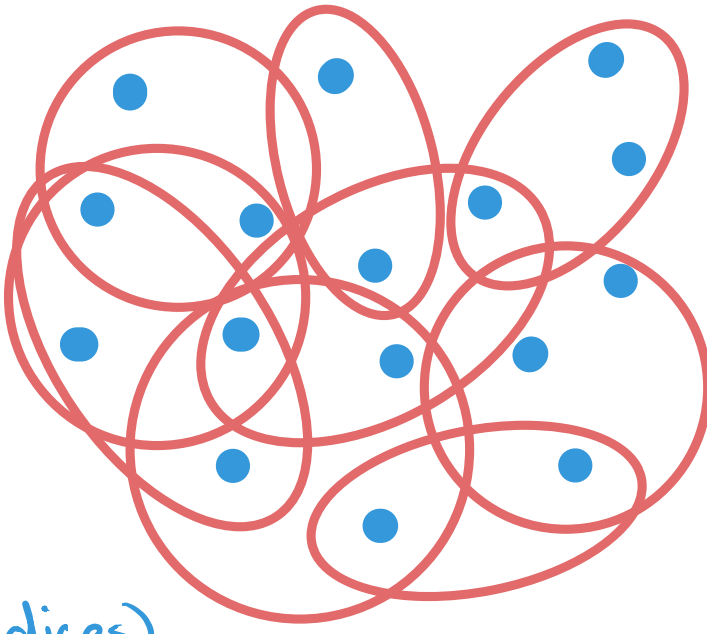
Greedy Algorithm



greedy algorithm repeatedly selects the set covers the most uncovered points

how good is the greedy algorithm?

(approximation ratio)



m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, \ldots, A_n \subseteq [m]$

Notation

$S \subseteq [n]$ interpreted as collection of sets (via indices)

$$f(S) \overset{\text{def}}{=} \left(\begin{array}{l} \#\text{ points covered by} \\ \bigcup_{i \in S} A_i \end{array} \right) = \left| \bigcup_{i \in S} A_i \right|$$

goal: max $f(S)$ over $S \subseteq [n]$ s.t. $|S| \leq k$

let $e_1, e_2, \ldots, e_k \in [n]$ be (indices of) sets taken by greedy in 1st, 2nd, 3rd, ... iteration

let $S_i = \{e_1, \ldots, e_i\}$ = sets taken by greedy after i iterations

$S_0 = \emptyset$, S_k = output of greedy

want to show:

$$f(S_k) \geq \boxed{??} \; OPT$$

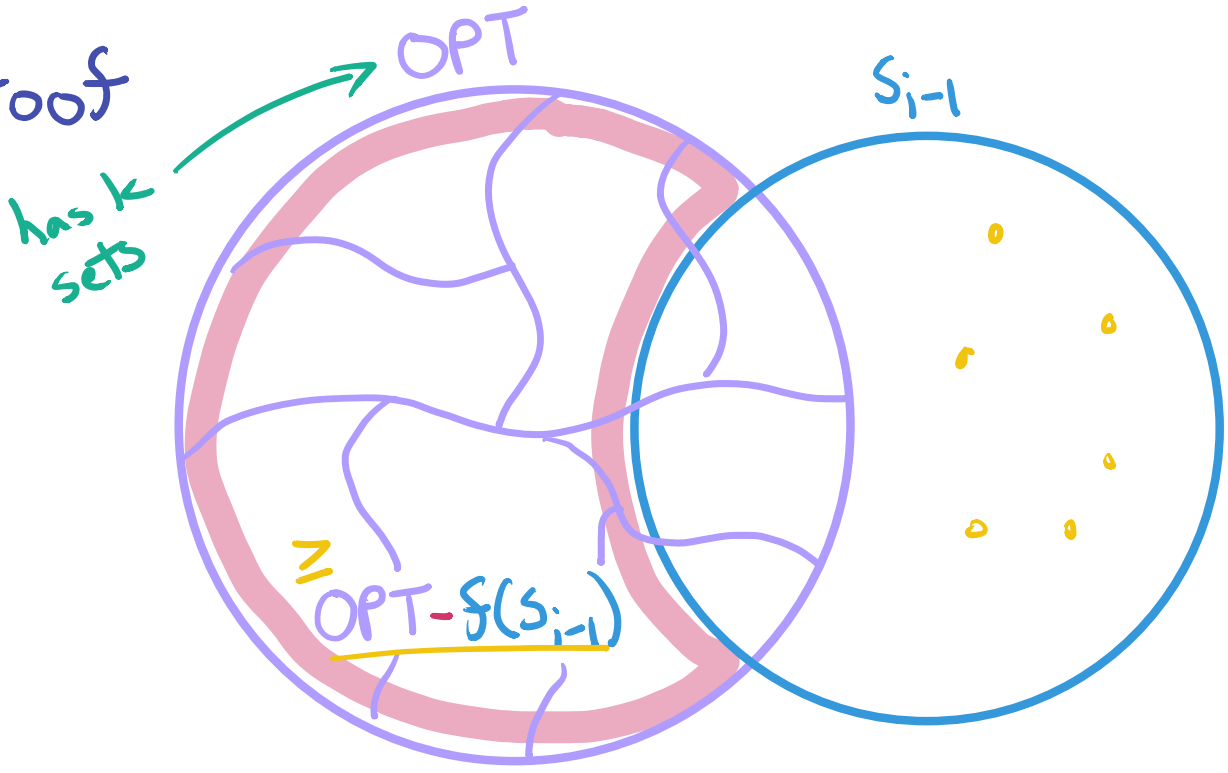
output

$f(S_k) \geq$ [ ] OPT

Key Lemma for $i = 1, \ldots, k$

improvement: $f(S_i) - f(S_{i-1})$ $\geq \frac{1}{k}$ (room for improvement: $OPT - f(S_{i-1})$)

Proof



some set in OPT covers $\frac{1}{k-1}(OPT - f(S_{i-1}))$ improvement from greedy is at least as much as this set

Overall analysis

$f(1-\frac{1}{k})OPT$ $\quad$ $f(1-\frac{1}{k})OPT$

improvement: $f(S_i) - f(S_{i-1})$ $\quad$ room for improvement: $OPT - f(S_{i-1})$

$$f(S_i) - f(S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$$

room for improvement after i: $OPT - f(S_i)$ $\quad$ room for improvement after $i-1$: $OPT - f(S_{i-1})$

$$OPT - f(S_i) \leq (1-\frac{1}{k})(OPT - f(S_{i-1}))$$

"room for improvement" decays exponentially

"room for improvement"

$$OPT - f(S_0) = OPT$$

$\leq (1-\frac{1}{k})$: $OPT - f(S_1) \leq (1-\frac{1}{k})\,OPT$

$(1-\frac{1}{k})$: $OPT - f(S_2) \leq (1-\frac{1}{k})^2\,OPT$

$OPT - f(S_3) \leq (1-\frac{1}{k})^3\,OPT$

$\vdots$

$$OPT - f(S_k) \leq (1-\frac{1}{k})^k\,OPT$$

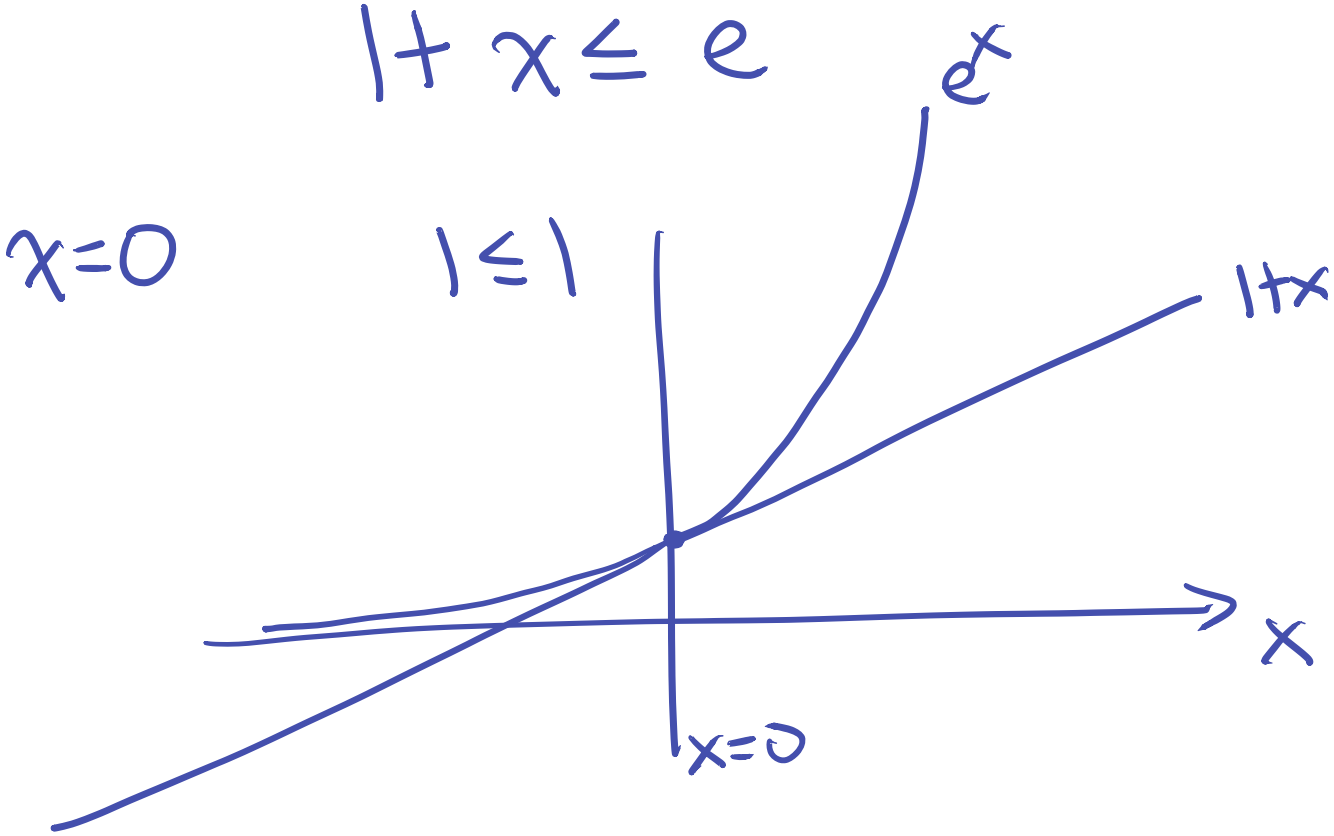
$$\leq e^{-1}\,OPT \quad (1+x \leq e^x)$$

$(1-\frac{1}{k})^k \leq (e^{-\frac{1}{k}})^k$

$\approx .68$

for all x,

$$1 + x \leq e^x$$



$$OPT - f(S_k) \leq e^{-1} OPT$$

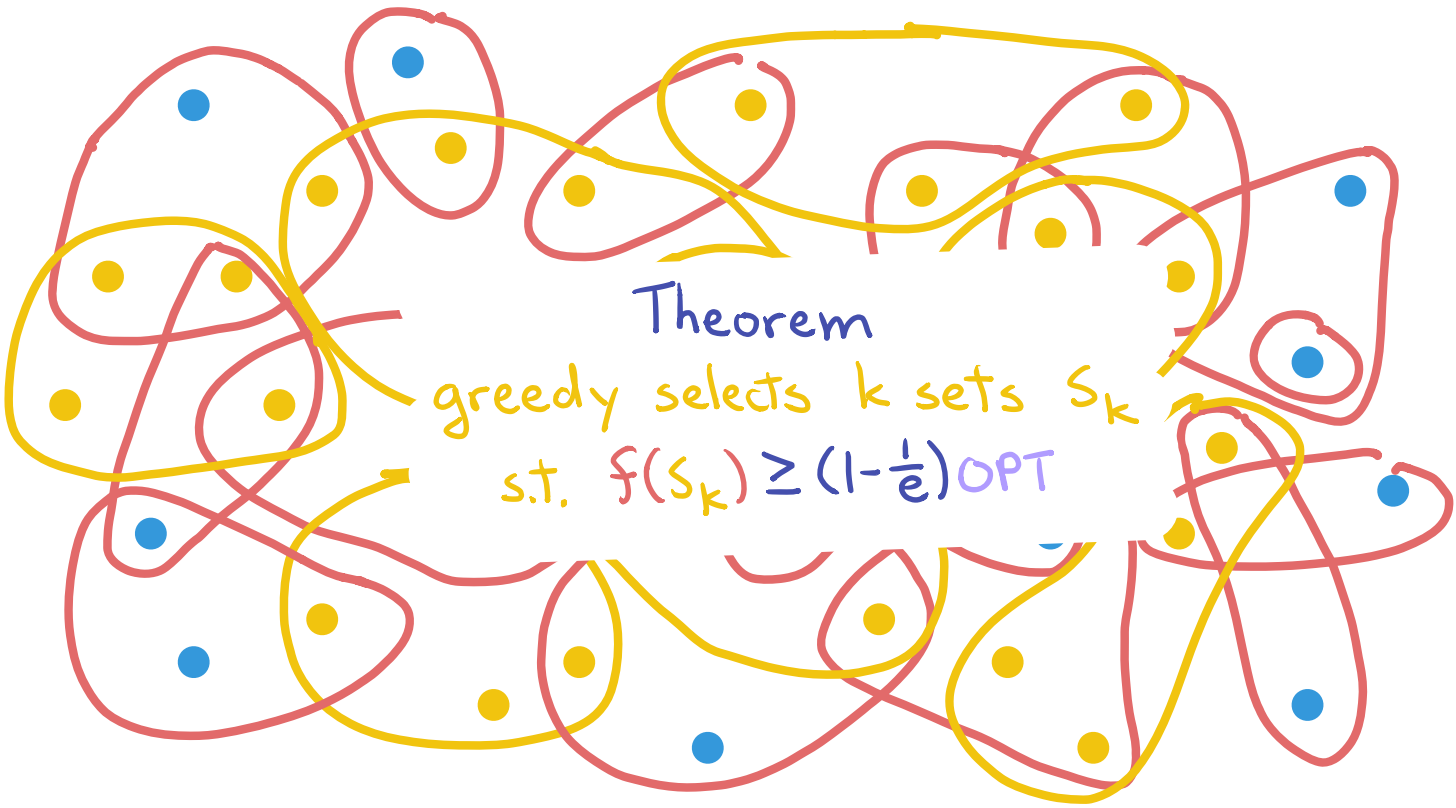
$$f(S_k) \geq \boxed{(1-e^{-1})} OPT$$

$\approx .6321$

$f(S) =$ ① monoton. incr. in S

② decreasing marginal returns

↑ monotone submodular



Part II: LP duality

Notation

math prog.

$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ $\quad x, y \in \mathbb{R}^n$

A^T = transpose of A $\quad A \in \mathbb{R}^{m \times n}$

def. by $\langle y, Ax \rangle = \langle A^T y, x \rangle$ $\quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$

$x \leq y$ means $x_i \leq y_i$ for all i $\quad x, y \in \mathbb{R}^n$

$(A^T)_{ij} = A_{ji}$

Integer Programs

$\min/\max \quad \sum_{j=1}^{n} c_j x_j$ — linear objective

over $x \in \mathbb{Z}^n$ — integral variables

s.t. $A_1 x \le b_1$

$A_2 x = b_2$

$A_3 x \ge b_3$

linear constraints

$A_1 x \le b_1$

for fixed vectors c, b_1, b_2, b_3 and matrices A_1, A_2, A_3

$$A_1 x \le b_1$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & \ddots & & \end{bmatrix} \begin{bmatrix} x \\ \vdots \end{bmatrix} \le \begin{bmatrix} b_1 \\ \vdots \end{bmatrix}$$

$$\forall i \quad A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n \le b_i$$

Linear programs

similar to integer programs, except removing integrality constraints

$$\min/\max \quad \sum_{j=1}^{n_1} c_j x_j$$

linear objective

over $x \in \mathbb{R}^n$

linear ~~integral~~ variables

s.t. $A_1 x \leq b_1$

$A_2 x = b_2$

$A_3 x \geq b_3$

linear constraints

for fixed vectors c, b_1, b_2, b_3 and matrices A_1, A_2, A_3

what's the difference?

Discrete optimization vs Continuous optimization

Integer Programs vs Linear Programs

NP-Hard

polynomial time!

Linear Programs are in P (!!)

(weakly polynomial)

1947 - Simplex algorithm
- Dantzig

50

(not polytime but practical)

1979 - Ellipsoid algorithm
- Khachiyan

high dim. geom
John's Ellipsoid theorem

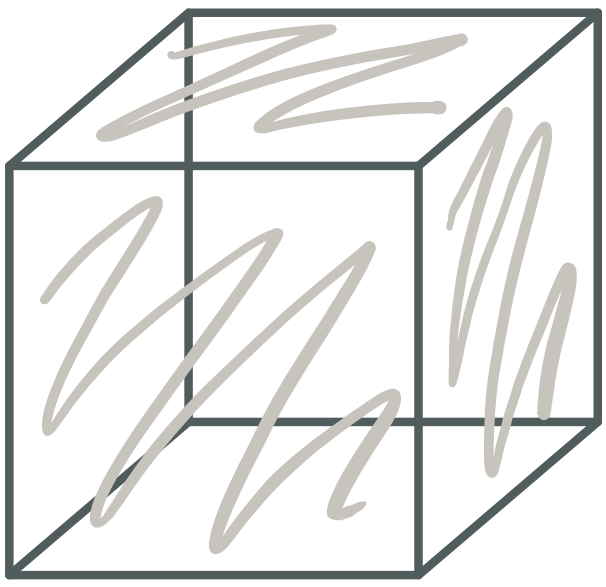
polytime

1984 - Interior point algorithm
- Kharmarkar

Newton's method

$$\min/\max \sum_j c_j x_j$$
$$\text{over } x \in \mathbb{R}^n$$
$$\text{s.t. } A_1 x \leq b_1$$
$$A_2 x = b_2$$
$$A_3 x \geq b_3$$

LP solver



optimal solution $x \in \mathbb{R}^n$

Today, special cases of

Packing and Covering

Packing LP

maximize $\langle b, x \rangle$

over $x \in \mathbb{R}^n_{\geq 0}$

s.t. $Ax \leq c$

packing (s,t) paths

Covering LP

minimize $\langle c, y \rangle$

over $y \in \mathbb{R}^m_{\geq 0}$

s.t. $A^T y \geq b$

cover (s,t) paths

(s,t)-cuts

$A \in \mathbb{R}^{m \times n}_{\geq 0}$, $b \in \mathbb{R}^n_{\geq 0}$, $c \in \mathbb{R}^m_{\geq 0}$

dual

LP Duality Theorem

dual

Packing LP

maximize $\langle b, x \rangle$

over $x \in \mathbb{R}^n_{\geq 0}$

s.t. $Ax \leq c$

$=$

Covering LP

minimize $\langle c, y \rangle$

over $y \in \mathbb{R}^m_{\geq 0}$

s.t. $A^T y \geq b$

$A \in \mathbb{R}^{m \times n}_{\geq 0}$, $b \in \mathbb{R}^n_{\geq 0}$, $c \in \mathbb{R}^m_{\geq 0}$

"(P)"

"(C)"

Lemma $(P) \leq (C)$

LP Duality Theorem

Packing LP		Covering LP
maximize $\langle b, x \rangle$	$=$ (equal)	minimize $\langle c, y \rangle$
over $x \in \mathbb{R}^n_{\geq 0}$		over $y \in \mathbb{R}^m_{\geq 0}$
s.t. $Ax \leq c$		s.t. $A^T y \geq b$

let x soln (P)

$Ax \leq c$

let y soln (C)

$A^T y \geq b$

$$\langle b, x \rangle \leq \langle A^T y, x \rangle = \langle y, Ax \rangle \leq \langle y, c \rangle$$

hard: $(C) \leq (P)$

Normalize $\underline{b}, \underline{c} = \mathbb{1}$

LP Duality Theorem

Packing LP

maximize $\langle \mathbb{1}, x \rangle$

over $x \in \mathbb{R}^n_{\geq 0}$

s.t. $Ax \leq \mathbb{1}$

(equal)

Covering LP

minimize $\langle \mathbb{1}, y \rangle$

over $y \in \mathbb{R}^m_{\geq 0}$

s.t. $A^T y \geq \mathbb{1}$

$A \in \mathbb{R}^{m \times n}_{\geq 0}$, $b \in \mathbb{R}^n_{\geq 0}$, $c \in \mathbb{R}^m_{\geq 0}$

"(P)"

"(C)"

goal: show $(P) \geq (C)$

if $Ax \leq \mathbb{1}$

$\langle p, Ax \rangle \leq 1$

$x \rightarrow$

LP Duality Theorem

Packing LP		Covering LP
maximize $\langle \mathbb{1}, x \rangle$	$=$ (equal)	minimize $\langle \mathbb{1}, y \rangle$
over $x \in \mathbb{R}^n_{\geq 0}$		over $y \in \mathbb{R}^m_{\geq 0}$
s.t. $Ax \leq \mathbb{1}$		s.t. $A^T y \geq \mathbb{1}$

Lagrangian Relaxations

$$\Delta^m = \{ p \in \mathbb{R}^m_{\geq 0} : p_1 + \cdots + p_m = 1 \}$$

(i.e, distributions over $[m]$)

for $p \in \Delta^m$, define relaxation

$$\max \langle \mathbb{1}, x \rangle \text{ over } x \in \mathbb{R}^{(n)}_{\geq 0}$$

$$\text{s.t. } \langle p, Ax \rangle \leq \langle p, \mathbb{1} \rangle = 1$$

(L)

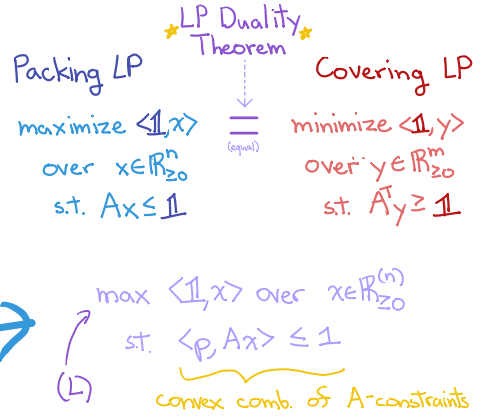
convex comb. of A-constraints

$$\langle p, Ax \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i A_{ij} x_j = \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} A_{ij} x_j \right)$$

$(L) \geq (P)$ (why?)

$(P) \leq (C)$

Lemma $(L) \geq (C)$



Fix $p \in \Delta^m$.

$\lambda > (L)$.

claim: λp is a feasible soln to (C)

(L) is so, then

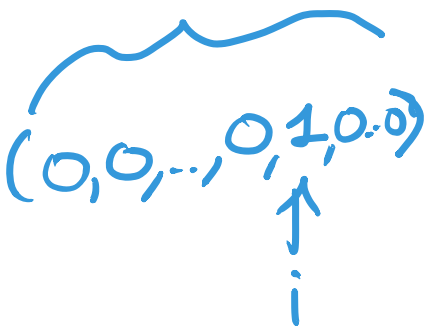
$\lambda \langle \mathbb{1}, p \rangle = \langle \mathbb{1}, \lambda p \rangle \geq (C)$

so $\lambda \geq (C)$ $\forall$ $\lambda > (L)$.

$\lim \lambda \downarrow (L) \Rightarrow (L) \geq (C)$

claim: $\lambda p \in (C)$

Fix $i \in [m]$. $x = \lambda e_i$.



$\langle \mathbb{1}, x \rangle = \lambda > (L)$

$\langle p, Ax \rangle \geq 1$

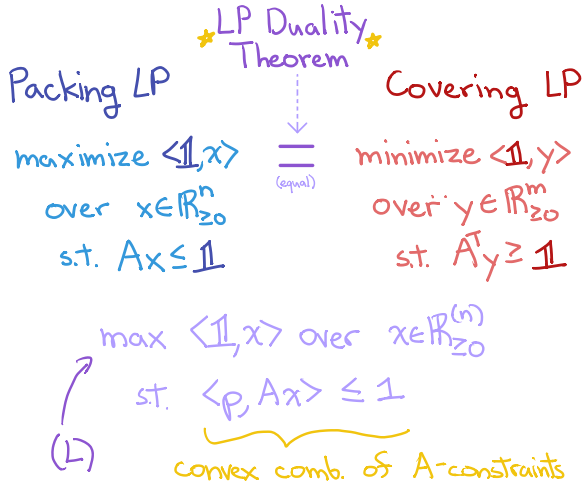
$\langle p, Ax \rangle = \langle p, A(\lambda e_i) \rangle = \lambda \langle p, Ae_i \rangle$

$= \lambda \langle A^T p, e_i \rangle = \lambda (A^T p)_i$

we know:

$(P) \leq (C) \leq (L)$

$(L) \geq (C)$ for all Lagrangian relaxations (L) of (P)



Partition Function (soft-max)

$$\pi(x) = \log\left(\sum_{i=1}^{m} e^{x_i}\right)$$

$$\max_i x_i = \log(e^{x_i})$$

$$\leq \log\left(\sum_{j=1}^{m} e^{x_j}\right) = \pi(x)$$

$$\leq \log(m \cdot \max_i e^{x_i})$$

$$= \frac{1}{\eta}\log(m) + \max_i e^{x_i}$$

$$\pi_i'(x) = \frac{e^{x_i}}{\sum_{j=1}^{m} e^{x_j}} \qquad \pi'(x) \in \Delta^m$$

LP Duality Theorem

Packing LP

maximize $\langle \mathbb{1}, x \rangle$
over $x \in \mathbb{R}^n_{\geq 0}$
s.t. $Ax \leq \mathbb{1}$

$=$ (equal)

Covering LP

minimize $\langle \mathbb{1}, y \rangle$
over $y \in \mathbb{R}^m_{\geq 0}$
s.t. $A^T y \geq \mathbb{1}$

(L) $\max \langle \mathbb{1}, x \rangle$ over $x \in \mathbb{R}^{(n)}_{\geq 0}$
s.t. $\langle p, Ax \rangle \leq 1$
convex comb. of A-constraints

$\pi(y) \times \max_i y_i$

Proof by "continuous algorithm"

$Ax \leq \mathbb{1}$ $\quad \max_i (Ax)_i$

define $f(x) = \frac{1}{\eta} \pi(\eta Ax) \approx \frac{1}{\eta} \max_i \eta (Ax)_i$

$\eta = \frac{\log(m)}{\varepsilon}$

approach: $x : [0,1] \to \mathbb{R}^n$

define $x(t)$ as continuous function $\in \mathbb{R}^n$ over time

s.t.

MWU

$x(0) = \mathbb{0}$

$\frac{d}{dt} \langle \mathbb{1}, x(t) \rangle \geq (c)$

$x(1)$ $\quad \frac{d}{dt} f(x(t)) \leq 1$

$$\langle \mathbb{1}, x(1) \rangle = \langle \mathbb{1}, x(1) \rangle - \langle \mathbb{1}, x(0) \rangle$$

$$= \int_0^1 \frac{d}{dt} \langle \mathbb{1}, x(t) \rangle \, dt$$

$$\geq \int_0^1 (c) \, dt = (c)$$

$$\max_i (Ax(1))_i \approx f(x(1)) - f(x(0))$$

$$= \int_0^1 \frac{d}{dt} f(x(t)) \, dt$$

$$\leq \int_0^1 1 \, dt = 1$$

end: $t=1, \langle \mathbb{1}, x \rangle \geq (c)$

$$\max_i (Ax(1))_i \leq 1$$

$$\Rightarrow (P) \geq (c)$$

need to show, at time

$$\exists \quad x'(t) = \frac{d}{dt} x(t)$$

$$= z$$

$$\text{s.t.} \quad \langle \mathbb{1}, x'(t) \rangle \geq (c)$$

$$\frac{d}{dt} f(x(t)) = \langle f'(x), x'(t) \rangle \leq 1$$

$$= z$$

$$f(x) = \frac{1}{\eta} \pi(\eta A x)$$

$$f'(x) = A^T \pi'(\eta A x)$$

$$\langle \mathbb{1}, z \rangle \geq (c) \qquad \langle f'(x), z \rangle \leq 1$$

$$= \langle A^T \pi'(\eta A x(t)), z \rangle$$

$$\max \langle \mathbb{1}, z \rangle \quad \text{s.t.} \quad \langle \pi'(\eta A x(t)), Az \rangle \leq 1$$

$$p \in \Delta^m$$

Lagrangian relaxation (L)

(L) ≥ (C)

HW K, MCQ last one

MCQ S

E[# rounds

0
1
-ε
+ε
λ*